Math 2E Quiz 9 Morning - May 26th

Please write your name and ID on the front.

Show all of your work, and simplify all your answers. *There is a question on the back side.

1. Recall that $\nabla \bullet (\nabla \times \mathbf{F}) = 0$, this is always zero (when the components of $\mathbf{F}$ have continuous 2nd order partial derivatives). Consider $\mathbf{F}(x, yz) = < z\cos y,\ xz\sin y,\ x\cos y >$.

(a) [4pts] Compute $\nabla \times \mathbf{F}$. Is $\mathbf{F}$ conservative?

$$\nabla \times \vec{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z\cos y & xz\sin y & x\cos y \end{bmatrix} =$$

$$= (-x\sin y - x\sin y)\hat{i} + (\cos y - \cos y)\hat{j} + (z\sin y - (-z\sin y))\hat{k}$$

+1 +1 +1 +1

$$= -2x\sin y\,\hat{i} + 0\hat{j} + 2z\sin y\,\hat{k}\ ; \quad \nabla \times \vec{F} \neq \vec{0} \text{ so, not conservative.}$$

(b) [2pts] Verify that $\nabla \bullet (\nabla \times \mathbf{F}) = 0$ by taking the divergence of your answer in (a).

$$\nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot < -2x\sin y, 0, 2z\sin y >$$ +1

$$= -2\sin y + 2\sin y = 0$$ +1 ✓

(c) [4pts] Prove that $\nabla \bullet (f\nabla g) = \nabla f \bullet \nabla g + f\nabla^2 g$.
Here, f, g are smooth scalar functions on $\mathbb{R}^3$ and $\nabla^2 g = g_{xx} + g_{yy} + g_{zz} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}$.
(∇^2 is called the "Laplacian" and it can also be denoted by Δ).

Pf: $\nabla \cdot (f\nabla g) = \nabla \cdot < fg_x, fg_y, fg_z >$

$$= \frac{\partial}{\partial x}(fg_x) + \frac{\partial}{\partial y}(fg_y) + \frac{\partial}{\partial z}(fg_z)$$ +1

$$= f_x g_x + f g_{xx} + f_y g_y + f g_{yy} + f_z g_z + f g_{zz}$$ +1

$$= f(g_{xx} + g_{yy} + g_{zz}) + (f_x g_x + f_y g_y + f_z g_z)$$ +1

$$= f(\nabla^2 g) + \nabla f \cdot \nabla g$$ ✓ +1 ▨

(Note: This S is also like $y = \sqrt{2xz}$ if we eliminate the parameter.)

2. Consider the surface S parameterized by $\mathbf{r}(u,w) = < u^2,\ uw,\ \frac{w^2}{2} >,\ 0 \le u \le 1,\ 1 \le w \le 2.$

(a) [3pts] Compute $\mathbf{r_u} \times \mathbf{r_w}$.

+1 $\vec{r}_u = < 2u, w, 0 >$

+1 $\vec{r}_w = < 0, u, w >$

$\Rightarrow \vec{r}_u \times \vec{r}_w = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & w & 0 \\ 0 & u & w \end{bmatrix}$

$= \boxed{w^2 \hat{i} - 2uw\hat{j} + 2u^2 \hat{k}}$

+1

(b) [3pts] Find the equation of the tangent plane at the point $(\frac{1}{4}, 1, 2)$. Assume $u, w \ge 0$.

At $(\frac{1}{4}, 1, 2)$, $u^2 = 1/4 \rightarrow u = 1/2$

$4w = 1$ +1

$\frac{w^2}{2} = 2 \rightarrow w = 2$

$\Rightarrow \vec{r}_u \times \vec{r}_w \Big|_{\substack{\text{At} \\ u=1/2 \\ w=2}} = < 4, -2, \frac{1}{2} >$. +1

Normal Vector!

Thus, plane eqn $\rightarrow \boxed{4(x-\frac{1}{4}) - 2(y-1) + \frac{1}{2}(z-2) = 0}$ +1

(c) [4pts] Compute the surface area of S on the given domain $0 \le u \le 1,\ 1 \le w \le 2$.

Defn: $\text{Area}(S) = \iint_{\mathcal{D}(u,w)} |\vec{r}_u \times \vec{r}_w| \, dA$. From (a), we have that:

$|\vec{r}_u \times \vec{r}_w| = \sqrt{w^4 + 4u^2w^2 + 4u^4} = \sqrt{(w^2 + 2u^2)^2} = w^2 + 2u^2.$ +1

Thus, $\text{Area}(S) = \int_{u=0}^{1} \int_{w=1}^{2} (w^2 + 2u^2)\, dw\, du$ +1

$= \int_{u=0}^{1} \frac{w^3}{3} + 2u^2 w \Big|_{w=1}^{2} du = \int_{u=0}^{1} \left(\frac{8}{3} + 4u^2 - \frac{1}{3} - 2u^2\right) du$

$= \int_0^1 \left(\frac{7}{3} + 2u^2\right) du = \frac{7}{3}u + \frac{2u^3}{3} \Big|_0^1$

$= \frac{7}{3} + \frac{2}{3} = \frac{9}{3} = \boxed{3 \text{ units}^2}$ +2